UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH7202**
ASSESSMENT PATTERN	:	**MATH7202A**
MODULE NAME	:	**Algebra 4: Groups and Rings**
DATE	:	**19-May-09**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2008/09-MATH7202A-001-EXAM-169

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define $\mathrm{ord}(g)$ when g is an element in a finite group G.

   Show that if each $g \in G$ has $\mathrm{ord}(g) \leqslant 2$ then for some $n \geqslant 1$

   $$G \cong \underbrace{C_2 \times \ldots \times C_2}_{n}.$$

   Define the *kernel* $\mathrm{Ker}(\varphi)$ and *image* $\mathrm{Im}(\varphi)$ of a group homomorphism $\varphi : G \to H$.

   State and prove a relationship which holds between $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$.

   Deduce that if $x \in G$ then $\mathrm{ord}\ \varphi(x)$ divides both $|G|$ and $|H|$.

   Describe explicitly all homomorphisms $\varphi : C_{10} \to C_{35}$.

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   (a) the *orbit*, $\langle x \rangle$, of x ;

   (b) the *stability group* G_x of x .

   Explain, with proof, what is meant by the *class equation* of the action in both its set-theoretic and numerical forms.

   Let D_6^* denote the binary dihedral group of order 12 and consider its action on itself by *conjugation* $\circ : D_6^* \times D_6^* \to D_6^*$; $g \circ h = ghg^{-1}$. Give an explicit description of

   i) the orbits in this action ;

   ii) the stability subgroup of a representative element in each orbit ;

   iii) both forms of the class equation.

3. Let p be a prime, and let G be a group of order p^n ($n \geqslant 1$) acting on a finite set X. Define the *fixed point set* X^G, and prove that $|X| \equiv |X^G| \pmod{p}$.

   Deduce that the centre $Z(G)$ of G is nontrivial.

   By means of a suitable action, show that $\binom{kp^n}{p^n} \equiv k \pmod{p}$ for any integer $k \geqslant 1$.

4. Let P, Q be subgroups of a group G; explain what is meant by saying that P *normalizes* Q. Moreover, in this case show that there is a group isomorphism

$$PQ/Q \cong P/(P \cap Q).$$

Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that $N_p \equiv 1 \pmod{p}$.

Deduce that if G is a group of order 205 then G has a normal subgroup of order 41.

Let $\alpha : C_{41} \to C_{41}$ be the automorphism given by $\alpha(x) = x^{10}$. Find the order of α in $\mathrm{Aut}(C_{41})$. Hence or otherwise describe all homomorphisms $h : C_5 \to \mathrm{Aut}(C_{41})$.

How many distinct groups of order 205 are there ? Justify your answer.

5. Let $\mathbb{F}$ be a field and let G be a finite subgroup of the multiplicative group $\mathbb{F}^*$. Prove that G is cyclic.

Show that $p(x) = x^2 + 2x + 4$ is irreducible over the field $\mathbb{F}_5$.

Let G denote the unit group $G = [\mathbb{F}_5[x]/(x^2 + 2x + 4)]^*$. By showing that $x + 4$ is a generator for G show explicitly that $G \cong C_n$ for some n, and state the value of n.

6. State and prove Eisenstein's criterion.

In each case below, decide whether or not the given polynomial is irreducible over $\mathbb{Q}$; moreover, if the polynomial is not irreducible, give its complete factorization into $\mathbb{Q}$-irreducible factors.

(i) $x^4 - x^3 - 3x^2 + 5x + 4$;

(ii) $x^8 - 22x^4 - 75$.

END OF PAPER